Aggregating agents with opinions about different propositions

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Abstract

There are many reasons we might want to take the opinions of various individuals and aggregate them to give the opinions of the group they constitute. If all the individuals in the group have probabilistic opinions about the same propositions, there is a host of aggregation functions we might deploy, such as linear or geometric pooling. However, there are also cases where different members of the group assign probabilities to different sets of propositions, which might overlap a lot, a little, or not at all. There are far fewer proposals for how to proceed in these cases, and those there are have undesirable features. I begin by considering four proposals and arguing that they don't work; then I'll describe my own proposal. In fact, my proposal breaks down into two proposals, each suited to a different purpose we might have when we aggregate. One purpose is *descriptive*, the other *normative*. In descriptive cases, we aggregate the individuals' credences in order to provide a compressed summary description of their opinions; in normative cases, we aggregate the credences to provide an account of the group's opinion, where the group is in some sense treated as an entity in its own rights.

1 Introduction

There are many reasons we might want to take the opinions of various individuals and aggregate them to give the opinions of the group they constitute. They might be demographic modellers, and we wish to summarise their views for policymakers. Or they might be ice sheet modellers and we wish to aggregate the probabilities they assign to various future sea level scenarios in order to include these in our global climate models (Bamber & Aspinall, 2013; Bamber et al., 2019). We might be producing a textbook on the epidemiology of respiratory viruses, and we wish to present something that we might legitimately call the view of the scientific community (French, 1987, 2011). Or we might be the lead author on a scientific paper with many co-authors and we wish to ensure that the conclusions presented in the paper are genuinely those of the entire group of authors (Bright et al., 2017). Outside science, the individuals we wish to aggregate might be employees of a company or institution whose collective opinion we wish to assess in order to determine liability for some harm, such as the board members of tobacco, oil, or social media companies, or the senior management of a police force (Lackey, 2020). Or they might be superforecasters, renowned for the accuracy of their previous predictions of future political or sporting events, and we wish to learn what they, as a group, think about a forthcoming election or the future state of democratic institutions in a country (Tetlock & Gardner, 2015). And so on.

If all the individuals in the group have probabilistic opinions about the same propositions, there is a host of aggregation functions we might deploy. For instance, linear pooling takes the group's probability for a proposition to be the arithmetic mean of the probabilities that its members assign to that proposition. Or, to calculate the group's probabilities for the possible states of the world, geometric pooling takes, for each state, the geometric mean of the probabilities that its members assign to that state, and then normalizes the results to ensure the pooled probabilities for the possible states sum to one. And so on. Each of these methods has its own desirable and undesirable features, which have been explored extensively (Genest & Zidek, 1986; Dietrich & List, 2015).

However, there are also cases where different members of the group assign probabilities to different sets of propositions, which might overlap a lot, a little, or not at all. Indeed, unless the probabilities are elicited by asking the same roster of questions to each individual in the group, this is the most likely situation we encounter in the wild. For instance, if we learn the probabilities that the members of a group of superforecasters assign by looking at their betting behaviour in a prediction market, any individual will likely have entered bets on some propositions that others have ignored, and failed to enter bets on some propositions that others have considered. One might have considered the proposition that Apple's stock price will rise and the proposition that Microsoft's will, but not the proposition that Apple's will rise more than Microsoft's, while another considered Microsoft's stock price and the comparison with Apple, but not Apple's stock price (Osherson & Vardi, 2006). And if we glean the probabilities that academic experts in an area assign by looking at what they report in their scholarly publications, we will find the same thing: probabilities reported for some propositions, but not others. One climate scientist might assign a probability to sea levels rising by at least 60cm by 2100, but nothing more fine-grained, while another might assign probabilities to it rising by 60-80cm, 80-100cm, and more than 100cm by that date. As they are usually formulated, most aggregation functions don't cover these cases. In this paper, I explore how we might fill that gap.

In Section 2, I'll introduce the formal framework in which we'll explore our problem. In Sections 3-7, I'll consider four proposals and argue that they don't work. Some of these exist in the literature explicitly as an answer to our question; some exist as answers to different questions, but are naturally repurposed to address ours; and some are simply occur to us naturally when we consider the question. Because there is so little written on this question, I begin with these four unsatisfactory proposals partly in order to clear the ground. But we will also see that, by doing so, an alternative proposal suggests itself. In fact, two proposals suggest themselves, each suited to a different purpose we might have when we aggregate. One purpose is *descriptive*, the other *normative*. In descriptive cases, we aggregate the individuals' credences in order to provide a compressed summary description of their opinions. I treat these sorts of case in Section 9.1. In normative cases, we aggregate the credences to provide an account of the group's opinion, where the group is in some sense treated as an entity in its own rights. I treat these sorts of case in Section 9.2.

2 The formal framework

Let me begin by laying out the formal framework we'll be working within.

- *Individuals* Let's assume that there are n individuals whose opinions we wish to aggregate.

- *Propositions* Let $\mathcal{F}_i$ be the set of propositions to which individual i assigns subjective probabilities or degrees of belief, which we will call *credences* throughout. We might call $\mathcal{F}_i$ their *agenda* or *opinion set*. Let $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$ be the union of all the individuals' agendas. Throughout, we assume that each $\mathcal{F}_i$ is finite, and therefore $\mathcal{F}$ is finite too.
- *Possible states of the world* Let $\mathcal{W}$ be the set of possible worlds grained just finely enough to assign truth values to each proposition in $\mathcal{F}$. We might represent $\mathcal{W}$ as the set of classically consistent assignments of truth values to the propositions in $\mathcal{F}$. Since each $\mathcal{F}_i$ is finite and therefore $\mathcal{F}$ is finite, $\mathcal{W}$ is also finite. If a proposition X in $\mathcal{F}$ is true at world w in $\mathcal{W}$, we write $w \models X$.
- *Subjective probabilities/credences* Let P_i record the credences that individual i assigns to the propositions in $\mathcal{F}_i$. We'll call this their *credence function*. For X in $\mathcal{F}_i$, $P_i(X)$ is the credence that individual i assigns to X. It is at least 0 and at most 1. We assume that these credences functions are *coherent*: that is, if $\mathcal{F}_i^*$ is the smallest Boolean algebra that contains $\mathcal{F}_i$, then it is possible to extend each P_i to a credence function P_i^* on $\mathcal{F}_i^*$ that satisfies the probability axioms—that is, P_i^* assigns credence 1 to the tautology, 0 to the contradiction, and the credence it assigns to a finite disjunction of incompatible propositions is the sum of the credences it assigns to the disjuncts.
- *Aggregation functions* An aggregation function Δ takes a sequence of n credence functions, $P_1, \ldots, P_n$, where P_i assigns credences to the propositions in $\mathcal{F}_i$, and returns a credence function $\Delta(P_1, \ldots, P_n)$, which assigns credences to the propositions in $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$.

Existing accounts of judgment aggregation deal with the particular case in which $\mathcal{F}_1 = \ldots = \mathcal{F}_n = \mathcal{F}$. They often also assume that $\mathcal{F}$ is a Boolean algebra. In such cases, for every world w in $\mathcal{W}$, there is a proposition in $\mathcal{F}$ that is true at w and only at w—these are sometimes called the *atoms* of the Boolean algebra $\mathcal{F}$. We abuse notation and write w for that proposition. We can then define linear and geometric pooling as follows:

Linear pooling For $P_1, \ldots, P_n$ defined on the same agenda $\mathcal{F}$, if X is in $\mathcal{F}$, then

$$\Delta_{\mathrm{LP}}(P_1, \ldots, P_n)(X) = \frac{1}{n} \sum_{i=1}^{n} P_i(X)$$

That is, the credence that the linear pool of $P_1, \ldots, P_n$ assigns to a possible world is the arithmetic mean of the credences that each P_i assigns to it.

Geometric pooling For $P_1, \ldots, P_n$ defined on the same agenda $\mathcal{F}$, which is a Boolean algebra, if w is in $\mathcal{W}$, then

$$\Delta_{\mathrm{GP}}(P_1, \ldots, P_n)(w) = \frac{\sqrt[n]{\prod_{i=1}^{n} P_i(w)}}{\sum_{w' \in \mathcal{W}} \sqrt[n]{\prod_{i=1}^{n} P_i(w')}}$$

And, for X in $\mathcal{F}$,

$$\Delta_{\mathrm{GP}}(P_1, \ldots, P_n)(X) = \sum_{w \models X} \Delta_{\mathrm{GP}}(P_1, \ldots, P_n)(w)$$

That is, the credence that the geometric pool of $P_1, \ldots, P_n$ assigns to a possible world is the normalized geometric mean of the credences that each P_i assigns to it; and the credence it assigns to a proposition is the sum of the credences it assigns to the worlds at which the proposition is true.

A few things to note:

- If each P_i is coherent, so is their linear pool.
- Whether or not each P_i is coherent, their geometric pool is coherent.
- Linear pooling is defined directly for each proposition in $\mathcal{F}$; as a result, we need not assume that $\mathcal{F}$ is a Boolean algebra.
- Geometric pooling is defined first for the states of the world in $\mathcal{W}$, and then for each proposition in $\mathcal{F}$; as a result, we must assume that $\mathcal{F}$ is a Boolean algebra.

In this paper, we ask: how should we aggregate in other cases? That is, how should we aggregate when two individuals have different agendas; that is, when $\mathcal{F}_i \neq \mathcal{F}_j$ for some individuals i and j?

In the following four sections, we consider different answers to this question. None of them work. We consider them partly to situate our proposal within the literature and clear the ground, but also because solving the problem that rules out the first two proposals motivates the account that we will go on to give in the remainder of the paper. The third proposal also attempts to solve that problem. It fails for a different reason, but one that is equally illuminating. Those impatient to hear about our solution can skip to Section 9.

3 Extending linear and geometric pooling

As we saw in the previous section, linear and geometric pooling are only defined in the special case in which $\mathcal{F}_1 = \ldots = \mathcal{F}_n = \mathcal{F}$; moreover, geometric pooling requires that $\mathcal{F}$ is a Boolean algebra. But perhaps we might generalize them so that they apply when $\mathcal{F}_i \neq \mathcal{F}_j$ for some individuals i and j?

For instance, suppose $\{X, Y, Z\}$ is a three-cell partition, and suppose the first of two individuals assigns credences to X, Y, and Z, so that $\mathcal{F}_1 = \{X, Y, Z\}$, while the second assigns credences only to X and Y, so that $\mathcal{F}_2 = \{X, Y\}$. Suppose their probability assignments are as follows:

	X	Y	Z
P_1	0.1	0.4	0.5
P_2	0.2	0.6	–

Then extending linear pooling to this case and taking the arithmetic means of the credences assigned to each gives:

	X	Y	Z
$\Delta_{\mathrm{LP}'}(P_1, P_2)$	0.15	0.5	0.5

But that's not coherent: the credences in X, Y, and Z sum to more than 1.

On the other hand, extending geometric pooling to this case and taking the geometric mean of the probabilities assigned to X, Y, and Z, and then normalizing, gives:

	X	Y	Z
$\Delta_{\mathrm{GP}'}(P_1, P_2)$	0.125	0.433	0.442

Obviously that is coherent, because geometric pooling requires us to normalise the geometric means; so the result will always be coherent.

Perhaps we should follow the lead of geometric pooling and do this for our extended version of linear pooling in such cases as well? So first we take the arithmetic means, and then we normalise the result. That would give:

	X	Y	Z
$\Delta_{\mathrm{LP}''}(P_1, P_2)$	0.1304	0.4347	0.4347

Unfortunately, both normalized extended linear pooling ($\Delta_{\mathrm{LP}''}$) and extended geometric pooling ($\Delta_{\mathrm{GP}'}$) violate a principle that I take to govern judgment aggregation in the cases we are considering, where the agendas of some of our individuals differ.

> **Dimension Insensitivity (DI)** If, for each individual i, there is a unique coherent credence function $P_i^\star$ defined on $\mathcal{F} = \bigcup_{i=1}^n \mathcal{F}_i$ that extends P_i, then $\Delta(P_1, \ldots, P_n) = \Delta(P_1^\star, \ldots, P_n^\star)$.[1]

The point is well illustrated by the example we've been considering in this section. While P_2 does not assign a credence to Z, it does assign credences to X and Y and together those determine the credence it would have to assign to Z in order to remain coherent—since X, Y, Z form a partition, it must assign 0.2. Dimension Insensitivity (DI) says that, in cases like this, where the probabilities that an individual assigns to the propositions in $\mathcal{F}_i$ determine the probabilities they must assign to the remaining propositions in $\mathcal{F}$, the result of aggregating the original probability assignments on $\mathcal{F}_1, \ldots, \mathcal{F}_n$ should be the same as the result of aggregating the probability functions on $\mathcal{F}$ that are obtained by filling in the gaps in the way that coherence requires. The idea is that, if the credences you have reported commit you to further credences, then adding those further credences explicitly shouldn't change the outcome of aggregating your credences with the credences of others. We will offer a partial accuracy-based justification of the principle in Section 8 below.

Thus, return to our case above:

	X	Y	Z
P_1	0.3	0.4	0.3
P_2	0.2	0.6	–
$P_2^\star$	0.2	0.6	0.2

So (DI) says that $\Delta(P_1, P_2) = \Delta(P_1, P_2^\star)$. But notice that neither normalized extended linear pooling ($\Delta_{\mathrm{LP}''}$) nor extended geometric pooling ($\Delta_{\mathrm{GP}'}$) deliver this:

	X	Y	Z
$\Delta_{\mathrm{LP}''}(P_1, P_2)$	0.1304	0.4347	0.4347
$\Delta_{\mathrm{LP}}(P_1, P_2^\star)$	0.15	0.5	0.35
$\Delta_{\mathrm{GP}'}(P_1, P_2)$	0.125	0.433	0.442
$\Delta_{\mathrm{GP}}(P_1, P_2^\star)$	0.149	0.517	0.333

(DI) will cause problems for the proposal we consider in the following section as well. But before we move on to that, there is another problem with our attempt to extend linear

[1] $P_i^\star$ defined on $\mathcal{F}$ extends P_i defined on $\mathcal{F}_i \subseteq \mathcal{F}$ if $P_i^\star(X) = P_i(X)$ for all X in $\mathcal{F}_i$. That is, if the restriction of $P_i^\star$ to $\mathcal{F}_i$ is just P_i.

and geometric pooling to the case in which $\mathcal{F}_i \neq \mathcal{F}_j$ for some i, j. Suppose $\mathcal{F}_1 = \{X \vee Y\}$ and $\mathcal{F}_2 = \{Y \vee Z\}$, where again X, Y, and Z form a partition. And suppose P_1 assigns credences only to the proposition in $\mathcal{F}_1$, while P_2 assigns only to the proposition in $\mathcal{F}_2$. In particular,

	$X \vee Y$	$Y \vee Z$
P_1	0.2	–
P_2	–	0.3

Now, first try to apply the extended linear pooling operator, $\Delta_{\mathrm{LP}''}$. By averaging the credences in each proposition, we get:

	$X \vee Y$	$Y \vee Z$
$\Delta_{\mathrm{LP}'}$	0.2	0.3

But that is incoherent: the credences in $X \vee Y$ and $Y \vee Z$ must sum at least to 1. So now we need to normalize. But how to do this? To normalize a credence function, we need to know the credences it assigns to the possible worlds. But in this case, we don't know that. So $\Delta_{\mathrm{LP}''}(P_1, P_2)$ is undefined. And of course the same fate befalls $\Delta_{\mathrm{GP}'}(P_1, P_2)$: indeed, it can't even get started, since it is defined initially on possible worlds, and then only at the second stage on logically weaker propositions.

4 A concern about Dimension Insensitivity

You might worry that it is not reasonable to demand that our aggregation rule satisfy (DI). After all, it seems that there are cases in which, when an individual i comes to consider a proposition to which they do not currently assign a credence, the process of considering it and assigning it a credence leads them to change the credences they already assign. This is the problem discusses in the literature on awareness growth (Karni & Vierø, 2013; Wenmackers & Romeijn, 2016; Bradley, 2017; Steele & Stefánsson, 2021; Mahtani, ta). A standard example is when the individual becomes aware of a possibility that they hadn't considered before. For instance, I might assign credences only to the propositions *It will rain tomorrow* and *It will be sunny tomorrow*, and assign credence 50% to each, but then come to consider a third possibility, namely, *It will be misty tomorrow*; and that might lead me to reduce my credence in the original two propositions in order to assign some credence to this new one.

I agree that this is possible and indeed rational. But it typically occurs when the individual learns a new concept or is made aware of a new possibility. In the sort of cases we're considering here this doesn't happen. Our examples in the introduction involve aggregating the opinions of reasonably cohesive collectives, such as the members of a scientific subdiscipline like glaciology or virology, or the executive employees of a corporation or institution. In these cases, it is reasonable to assume that the individuals in question have a shared set of concepts and a shared conception of the possibilities. So, when they do not assign a credence to a proposition, it is not because the possibility identified by that proposition or the concepts it contains would be entirely new to them. It is rather just that they are finite creatures, as we all are, and do not always assign credences to all propositions that their conceptual scheme determines. So, in the cases with which I am concerned here, it is reasonable to demand (DI).

5 The Coherent Approximation Principle

In Section 3, we saw that it is difficult to extend linear and geometric pooling so that they apply to the problem of aggregating credence functions defined on different agendas—that is, when $\mathcal{F}_i \neq \mathcal{F}_j$ for some i, j. In this section, we turn to one of the few treatments of the current problem from the literature. It is due to Daniel Osherson and Moshe Vardi (Osherson & Vardi, 2006).

In fact, Osherson and Vardi treat two problems at once. Not only do they not assume that the individuals to be aggregated assign credences to the same propositions; they also do not assume that those individuals assign coherent credences. So they seek an aggregation function that takes possibly incoherent credence functions over possibly different agendas and aggregates them into a coherent credence function on the union of the agendas. Their approach, which draws on the pioneering work of Konieczny & Pino-Pérez (1998, 1999), is distance-based. That is, we begin by identifying a measure of distance from one credence to another. We then take the aggregate of a set of credence functions to be the credence function for which the average of the sum of the distances from the credences that it assigns to the credences that the individuals assign is minimal. Osherson and Vardi consider two such measures of distance:

Absolute deviation For credences $0 \leq p, q \leq 1$,

$$\mathrm{AD}(p, q) = |p - q|$$

Squared deviation For credences $0 \leq p, q \leq 1$,

$$\mathrm{SD}(p, q) = |p - q|^2$$

And there are many others, including the popular Kullback-Leibler divergence:

Kullback-Leibler divergence For credences $0 \leq p, q \leq 1$,

$$\mathrm{KL}(p, q) = p \log \frac{p}{q} - p + q$$

We say that a measure $\mathfrak{d}$ of distance from one credence to another is a *divergence* if (i) $\mathfrak{d}(p, q) \geq 0$ for all $0 \leq p, q \leq 1$ and (ii) $\mathfrak{d}(p, q) = 0$ iff $p = q$. AD, SD, and KL are all divergences. Now, given a divergence $\mathfrak{d}$, here is Osherson and Vardi's aggregation function, where $\mathcal{P}_{\mathcal{F}}$ is the set of coherent credence functions on $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$:

Coherent Approximation Principle$_{\mathfrak{d}}$ (CAP$_{\mathfrak{d}}$) For P_i defined on $\mathcal{F}_i$,

$$\Delta^{\mathfrak{d}}_{\mathrm{CAP}}(P_1, \ldots, P_n) = \underset{P \in \mathcal{P}_{\mathcal{F}}}{\arg\min} \sum_{i=1}^{n} \sum_{X \in \mathcal{F}_i} \mathfrak{d}(P(X), P_i(X))$$

That is, $\Delta^{\mathfrak{d}}_{\mathrm{CAP}}(P_1, \ldots, P_n)$ is the coherent credence function for which the average of the sums of the divergences from its credences to the credences assigned by $P_1, \ldots, P_n$ is minimal.

In fact, if we wish the minimizer to be unique here, we must restrict the divergences that we use. For instance, recall our example from the previous section:

	X	Y	Z
P_1	0.1	0.4	0.5
P_2	0.2	0.6	–
$P_2^\star$	0.2	0.6	0.2

Then, if we use the absolute deviation to measure the distance from one credence to another—that is, if $\mathfrak{d} = \text{AD}$—then, providing $0.1 \leq P(X) \leq 0.2$, $0.4 \leq P(Y) \leq 0.6$, and $0.2 \leq P(Z) \leq 0.5$, P minimises the average distance to P_1 and $P_2^\star$. I presume for this reason, when Osherson writes about CAP again with different co-authors, they focus on squared deviation (Predd et al., 2008). We'll focus on squared deviation and Kullback-Leibler divergence for the moment.[2] Here are the results of aggregating P_1 and P_2 using $\Delta^{\text{SD}}_{\text{CAP}}$ and using $\Delta^{\text{KL}}_{\text{CAP}}$, and the results of aggregating P_1 and $P_2^\star$ using $\Delta^{\text{SD}}_{\text{CAP}}$ and using $\Delta^{\text{KL}}_{\text{CAP}}$.

	X	Y	Z
$\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2)$	0.113	0.462	0.425
$\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2^\star)$	0.15	0.5	0.35
$\Delta^{\text{KL}}_{\text{CAP}}(P_1, P_2)$	0.32	0.204	0.475
$\Delta^{\text{KL}}_{\text{CAP}}(P_1, P_2^\star)$	0.148	0.389	0.463

Since the first and second row differ, $\Delta^{\text{SD}}_{\text{CAP}}$ violates (DI); since the third and fourth row differ, $\Delta^{\text{KL}}_{\text{CAP}}$ violates (DI).

Now, you might try to save the Coherent Approximation Principle in one of two ways. First, you might seek a divergence $\mathfrak{d}$ for which $\Delta^{\mathfrak{d}}_{\text{CAP}}$ satisfies (DI). However, the following fact shows that this is impossible:

Proposition 1. *If $\mathfrak{d}$ is differentiable in its first argument, $\Delta^{\mathfrak{d}}_{\text{CAP}}$ violates (DI).*

(The proof is given in the Appendix.)

Second, you might think that the problem arises because the single credence assigned to Z is given exactly as much weight as the two credences assigned to X and the two credences assigned to Y. But it's easy to check that assigning twice as much weight to $\mathfrak{d}(P(Z), P_1(Z))$ as to $\mathfrak{d}(P(X), P_1(X))$ or $\mathfrak{d}(P(Y), P_1(Y))$ doesn't bring the Coherent Approximation Principle into agreement with (DI). For instance,

$$(\text{SD}(P(X), 0.1) + \text{SD}(P(X), 0.2)) + (\text{SD}(P(Y), 0.4) + \text{SD}(P(Y), 0.6)) + 2 \times \text{SD}(P(Z), 0.5)$$

is minimized among coherent functions at $P = (0.1, 0.45, 0.45)$, while

$$(\text{SD}(P(X), 0.1) + \text{SD}(P(X), 0.2)) + (\text{SD}(P(Y), 0.4) + \text{SD}(P(Y), 0.6)) + (\text{SD}(P(Z), 0.5) + \text{SD}(P(Z), 0.2))$$

is minimized among coherent credence functions at $P = (0.15, 0.5, 0.35)$.

We will return to (DI) below. So far, we have appealed only to its intuitive plausibility. In Section 8, we will compare the accuracy of the credences you obtain if you use it with the accuracy of the credences you obtain if you violate it in various ways.

[2]Pettigrew (2019) makes the same choice.

6 Aggregating the sets of coherent credence functions that extend the individuals' credence functions

Like the Coherent Approximation Principle, the third proposal we'll consider asks us to aggregate by minimizing the average distance from some representation of the individuals' opinions. But whereas CAP represents individual i by the precise credences they assign to the propositions in $\mathcal{F}_i$, the third proposal represents them by the imprecise credences they assign to the propositions in $\mathcal{F}$. That is, instead of representing individual i by the *single credence function* P_i, which is defined on $\mathcal{F}_i$, we represent them by the following *set of credence functions*, which are defined on $\mathcal{F}$, where $\mathcal{P}_{\mathcal{F}}$ is the set of coherent credence functions on $\mathcal{F}$, as above:

$$R_i = \{P : \mathcal{F} \to [0,1] \mid P \in \mathcal{P}_{\mathcal{F}} \ \&\ (\forall X \in \mathcal{F}_i)[P(X) = P_i(X)]\}$$

So, R_i is the set of coherent extensions of P_i to $\mathcal{F}$. And we aggregate $P_1, \ldots, P_n$ by aggregating $R_1, \ldots, R_n$. And we aggregate $R_1, \ldots, R_n$ by finding a credence function P that minimizes the average distance from P to the R_is, where the distance from P to R_i is the minimum distance between P and a member of R_i. For many divergences and many $P_1, \ldots, P_n$, this minimization problem will have a unique solution. In that case, we define:

$$\Delta^{\mathfrak{d}}_{\mathrm{MW}}(P_1, \ldots, P_n) = \arg\min_{P \in \mathcal{P}_{\mathcal{F}}} \sum_{i=1}^{n} \left(\min_{Q \in R_i} \sum_{X \in \mathcal{F}} \mathfrak{d}(P(X), Q(X)) \right)$$

I call this aggregation function $\Delta^{\mathfrak{d}}_{\mathrm{MW}}$, because this method for aggregating sets of probability functions is proposed by Martin Adamčík and George Wilmers (Adamčík & Wilmers, 2014; Wilmers, 2015). Seamus Bradley (2019) criticizes it as an aggregation rule for sets of probability functions that represent uncertainty in the imprecise credence framework. But his criticisms are less worrying when it is used to aggregate sets of probability functions that represent gaps in credal reporting, as we do here, so I won't repeat them.

It is easy to see that such an aggregation function will satisfy (DI). After all, if there is a unique coherent credence function $P^\star_i$, defined on $\mathcal{F}$, that extends P_i, which is defined on $\mathcal{F}_i$, then the set of coherent probability functions that extends P_i is the same as the set of coherent probability functions that extends $P^\star_i$—both contain only $P^\star_i$. That is:

$$\begin{aligned} R_i = \{P : \mathcal{F} \to [0,1] \mid P \in \mathcal{P}_{\mathcal{F}} \ \&\ (\forall X \in \mathcal{F}_i)[P(X) = P_i(X)]\} = \\ \{P^\star_i\} = \{P : \mathcal{F} \to [0,1] \mid P \in \mathcal{P}_{\mathcal{F}} \ \&\ (\forall X \in \mathcal{F}_i)[P(X) = P^\star_i(X)]\} = R^\star_i \end{aligned}$$

So this proposal does not suffer from the same problem as the previous two. But it does face a problem: it gives implausible answers in reasonably straightforward cases. For instance, suppose $\mathcal{F} = \{X, Y, Z\}$, where X, Y, and Z form a partition, and $\mathcal{F}_1 = \{X\}$ and $\mathcal{F}_2 = \{Y\}$. And suppose

	X	Y	Z
P_1	0.8	–	–
P_2	–	0.8	–

So:

- $R_1 = \{P \in \mathcal{P}_{\mathcal{F}} : P(X) = 0.8\}$
- $R_2 = \{P \in \mathcal{P}_{\mathcal{F}} : P(Y) = 0.8\}$

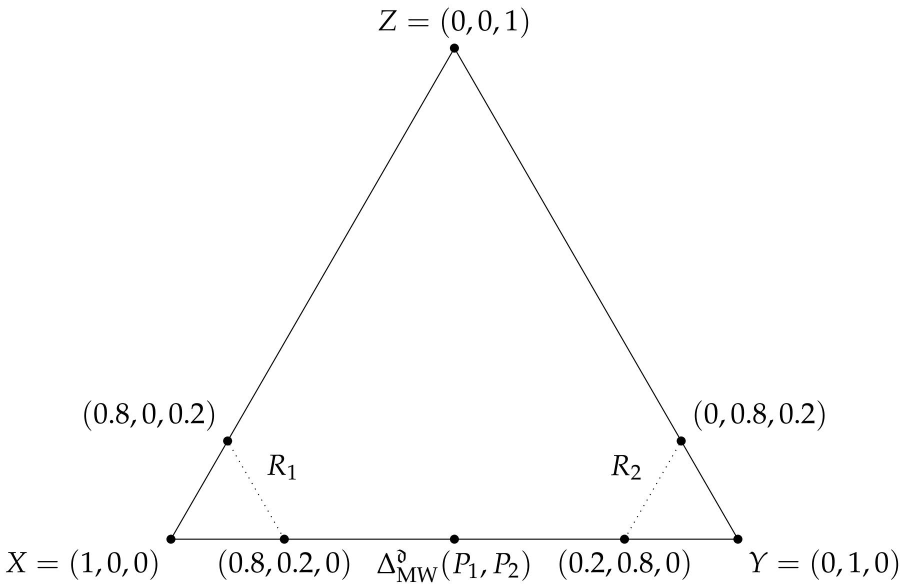

Figure 1: The barycentric plot of the 2-simplex with $(1,0,0)$ at bottom left, $(0,1,0)$ at bottom right, and $(0,0,1)$ at the top. The dotted lines represent R_1 and R_2, respectively. And the result of applying $\Delta^{\mathfrak{d}}_{\mathrm{MW}}$ to P_1 and P_2 is plotted.

We can illustrate these two sets of probabilities by plotting them within the three-dimensional simplex on a barycentric plot (see Figure 1). The problem is that, if $\mathfrak{d}$ is squared deviation (SD) or Kullback-Leibler divergence (KL), then $\Delta^{\mathfrak{d}}_{\mathrm{CAP}}(P_1, P_2)$ is as follows:

	X	Y	Z
$\Delta^{\mathfrak{d}}_{MW}(P_1, P_2)$	0.5	0.5	0

That is plotted on the simplex as well. The problem here is that this seems too extreme: it assigns no probability to Z, even though nothing in the opinions of either agent forces that. It is the same aggregate we would obtain if both agents were to assign zero credence to Z and fill in Y in such a way that they remained coherent. That is,

$$\Delta^{\mathfrak{d}}_{\mathrm{MW}}(P_1, P_2) = \Delta^{\mathfrak{d}}_{\mathrm{MW}}(P_1^\circ, P_2^\circ)$$

where

	X	Y	Z
P_1°	0.8	0.2	0
P_2°	0.2	0.8	0

7 Maximising entropy within the set of possible aggregates

Another proposal that arises naturally. Let

$$R_{LP} = \{\Delta_{\mathrm{LP}}(P'_1, \ldots, P'_n) : P'_i \in R_i\}$$

That is, R_{LP} is the set of linear pools of coherent extensions of the individuals' credence functions. Then let the aggregate of $P_1, \ldots, P_n$ be the credence function in R_{LP} with maximum entropy.[3] First, define the Shannon entropy of a probability function P defined over a set $\mathcal{W}$ of possible worlds as follows:

$$H(P) = -\sum_{w \in \mathcal{W}} P(w) \log P(w)$$

Then let

$$\Delta^{\mathrm{LP}}_{\mathrm{ME}}(P_1, \ldots, P_n) := \underset{P \in R_{LP}}{\arg\max}\, H(P)$$

The problem with this approach is that, again, it gives an implausible answer in the case we considered in the previous section. But this time it gives exactly the opposite implausible answer to the one above! Suppose

	X	Y	Z
P_1	0.8	–	–
P_2	–	0.8	–

Then

	X	Y	Z
$\Delta^{\mathrm{LP}}_{\mathrm{ME}}(P_1, P_2)$	0.4	0.4	0.2

Again, we illustrate this in a barycentric plot—see Figure 2.

The problem again is that this seems too extreme, albeit extreme in the opposite direction: it is the same aggregate we would obtain if the first agent were to assign zero credence to Y and the second were to assign zero credence to X.

8 On the accuracy of aggregation rules

In Sections 3 and 5, we criticized the extensions of linear and geometric pooling, $\Delta_{\mathrm{LP}''}$ and $\Delta_{\mathrm{GP}'}$, and the Coherent Approximation Principle, $\Delta^{\mathfrak{d}}_{\mathrm{CAP}}$, because they both violate (DI), the principle that says that, when there's a unique coherent extension of each credence function to the full algebra, aggregating those extensions should give the same result as aggregating the original credence functions. At that point, I merely appealed to the intuitive force of (DI)—I gave no further argument in its favour. But there is something to be said for aggregation rules that satisfy it, at least when they are compared with CAP.

Let's begin with a slight adaptation of the simple example from above:

	X	Y	Z
P_1	0.1	0.4	–
P_2	0.2	0.6	–
$P^\star_1$	0.1	0.4	0.5
$P^\star_2$	0.2	0.6	0.2

[3] Strictly speaking, this will only work if $\mathcal{F} = \bigcup_{i=1}^n \mathcal{F}_i$ contains $\mathcal{W}$. It's an interesting question how the proposal might be extended beyond this, perhaps by considering the extensions of each P_i not only to $\mathcal{F}$ but to $\mathcal{F}^*$, the smallest Boolean algebra that contains $\mathcal{F}$. But, as we will see, the proposal doesn't work, so I won't spend time on that.

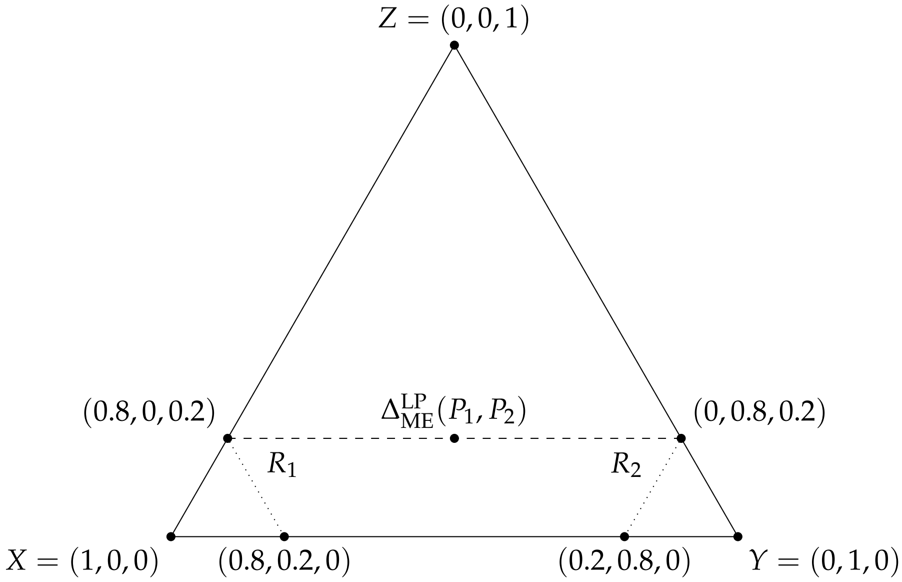

Figure 2: The barycentric plot of the 2-simplex with $(1,0,0)$ at bottom left, $(0,1,0)$ at bottom right, and $(0,0,1)$ at the top. The dotted lines represent R_1 and R_2, respectively. And the result of applying $\Delta^{\mathrm{LP}}_{\mathrm{ME}}$ to P_1 and P_2 is plotted.

(DI) says that aggregating P_1 and P_2 should give the same result as aggregating P_1 and $P_2^\star$, which should give the same result as aggregating $P_1^\star$ and P_2, which should give the same result as aggregating $P_1^\star$ and $P_2^\star$. That is, if Δ is our aggregation function,

$$\Delta(P_1, P_2) = \Delta(P_1, P_2^\star) = \Delta(P_1^\star, P_2) = \Delta(P_1^\star, P_2^\star)$$

But let's apply CAP using the squared deviation:

	X	Y	Z
$\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2)$	$\frac{12}{80}$	$\frac{40}{80}$	$\frac{28}{80}$
$\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2^\star)$	$\frac{15}{80}$	$\frac{43}{80}$	$\frac{22}{80}$
$\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2)$	$\frac{9}{80}$	$\frac{37}{80}$	$\frac{34}{80}$
$\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2^\star)$	$\frac{12}{80}$	$\frac{40}{80}$	$\frac{28}{80}$

Now, notice that $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2^\star)$ is the same as $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2)$, and both are the midpoint between $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2^\star)$ and $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2)$. That is,

$$\begin{aligned}
&\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2^\star) = \Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2) = \\
&\quad \tfrac{1}{2}(\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2^\star) + \Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2)) = \\
&\qquad \tfrac{1}{4}(\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2^\star) + \Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2) + \Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2^\star) + \Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2))
\end{aligned}$$

That is, when we include the credal assignment to Z that P_1 determines, but not the assignment that P_2 determines, $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}$ pulls the aggregate towards P_1 and away from P_2; and, mutatis mutandis, if we include the credal assignment to Z that P_2 determines, but not the one that P_1 determines. And, moreover, the pull is the same but in opposite directions in the two cases. So, when we average them, we obtain what we would have obtained if we'd left out both assignments to Z (and aggregated P_1 and P_2) or if we'd included both assignments to Z (and aggregated $P_1^\star$ and $P_2^\star$).

What does this tell us? Well, suppose our favoured aggregation function for those cases in which all individuals have the same opinion set (or agenda) is linear pooling; and suppose we extend that aggregation function in line with (DI). Then we can say that following in favour of our approach and against CAP. First, we note the following corollary of the Diversity Prediction Theorem (Galton, 1907; Page, 2007):

Theorem 2. *For any $\mathcal{F}$ and credence functions $Q, Q_1, \ldots, Q_n$ defined on $\mathcal{F}$,*

$$\sum_{X \in \mathcal{F}} \mathrm{SD}(\Delta_{\mathrm{LP}}(Q_1, \ldots, Q_n)(X), Q(X)) < \frac{1}{n} \sum_{i=1}^{n} \sum_{X \in \mathcal{F}} \mathrm{SD}(Q_i(X), Q(X))$$

This says that, for any credence function Q and any set of credence functions $Q_1, \ldots, Q_n$ all defined on the same set of propositions, the distance of the linear pool of $Q_1, \ldots, Q_n$ from Q is always less than the average distance of the Q_is from Q, when the distance between credences is measured using squared deviation.[4]

How does this help? Well, given a possible world w, let V_w be the credence function that assigns maximal credence to all propositions that are true at w and minimal credence to all propositions that are false at w: that is, $V_w(X) = 1$ if X is true at w, and $V_w(X) = 0$ if X is false at w. We might call V_w the omniscient credence function. It is natural to say that the ideal credence function for an individual to have at a world is the omniscient credence function at that world, and that a credence function is more accurate the closer it lies to that omniscient credence function. So we might say that the inaccuracy of a credence function Q_i at world w is the sum of the squared deviations between the credences it assigns and the credences that V_w assigns: we call this the Brier score of inaccuracy. So, if P is defined on $\mathcal{F}$,

$$\mathfrak{B}(P, w) = \sum_{X \in \mathcal{F}} (P(X) - V_w(X))^2$$

And we might think that a credence function is doing better, epistemically speaking, the greater its inaccuracy and the lower its Brier score. That is, P is better than Q at w just in case $\mathfrak{B}(P, w) < \mathfrak{B}(Q, w)$ (Brier, 1950; Pettigrew, 2016). Then, by Theorem 2,

Corollary 3. *For any $\mathcal{F}$, any world w, and any credence functions $Q_1, \ldots, Q_n$ defined on $\mathcal{F}$,*

$$\mathfrak{B}(\Delta_{\mathrm{LP}}(Q_1, \ldots, Q_n), w) < \frac{1}{n} \sum_{i=1}^{n} \mathfrak{B}(Q_i, w)$$

That is, the inaccuracy of the linear pool of $Q_1, \ldots, Q_n$ is less than the average inaccuracy of the Q_is.

Now, recall that the linear pool of $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2^\star)$, $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2)$, $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1, P_2^\star)$, and $\Delta^{\mathrm{SD}}_{\mathrm{CAP}}(P_1^\star, P_2)$ is just $\Delta_{\mathrm{LP}}(P_1^\star, P_2^\star)$. Then it follows that, for any world, the inaccuracy of $\Delta_{\mathrm{LP}}(P_1^\star, P_2^\star)$ at that

[4] Indeed, this generalizes to any convex divergence, but I'll focus on squared deviation here for the sake of concreteness.

world is less than the average inaccuracy of $\Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2^\star)$, $\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2)$, $\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2^\star)$, and $\Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2)$ at that world. That is,

$$\begin{aligned}&\mathfrak{B}\left(\Delta_{\text{LP}}(P_1^\star, P_2^\star), w\right) < \\ &\quad \tfrac{1}{4}\left(\mathfrak{B}\left(\Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2^\star), w\right) + \mathfrak{B}\left(\Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2^\star), w\right) + \right.\\ &\qquad\qquad \left.\mathfrak{B}\left(\Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2^\star), w\right) + \mathfrak{B}\left(\Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2^\star), w\right)\right)\end{aligned}$$

So aggregating in line with (DI) is more accurate than aggregating in line with CAP, at least in expectation and if you are equally likely to find yourself aggregating P_1 and P_2 as you are to find yourself aggregating P_1 and $P_2^\star$, or $P_1^\star$ and P_2, or $P_1^\star$ and $P_2^\star$.

Does this generalise beyond the specific case of P_1 and P_2? Yes, as the following theorem shows:

Theorem 4. *Suppose $\mathcal{F}, \mathcal{F}'$ are two sets of propositions and $\mathcal{F}' \subseteq \mathcal{F}$. Suppose P_1 is a credence function on $\mathcal{F}'$ and $P_1^\star$ is the unique coherent extension of P_1 to $\mathcal{F}'$; and suppose P_2 is a credence function on $\mathcal{F}'$ and $P_2^\star$ is the unique coherent extension of P_2 to $\mathcal{F}$. Then*

$$\Delta_{\text{LP}}(P_1^\star, P_2^\star) = \tfrac{1}{4}\left(\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2) + \Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2^\star) + \Delta^{\text{SD}}_{\text{CAP}}(P_1^\star, P_2) + \Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2^\star)\right)$$

Now, suppose you enter an aggregation task knowing only that the individuals will assign credences either to the propositions in $\mathcal{F}'$ or to the propositions in $\mathcal{F}$, where $\mathcal{F}' \subseteq \mathcal{F}$. Then, if we assume that there is no correlation between the particular credences the individuals assign and whether they assign them only to the propositions in $\mathcal{F}'$ or to the propositions in $\mathcal{F}$, then it is as likely that the group you wish to aggregate consists of P_1 and $P_2^\star$ as it is that it will consist of $P_1^\star$ and P_2, and as likely that it consists of P_1 and P_2, and as likely that it consists of $P_1^\star$ and $P_2^\star$. And if that's right then the expected inaccuracy of using a rule that respects (DI) and thus sets $\Delta(P_1^\star, P_2) = \Delta(P_1, P_2^\star) = \Delta_{\text{LP}}(P_1^\star, P_2^\star)$ is lower than the expected inaccuracy of using CAP, which will give each of $\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2^\star)$, $\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2^\star)$, $\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2^\star)$, and $\Delta^{\text{SD}}_{\text{CAP}}(P_1, P_2^\star)$ a probability of 25%.

9 Beyond Dimension Insensitivity

Dimension Insensitivity tells us how our aggregation function should work when, for each individual i, there is a unique coherent extension $P_i^\star$ of P_i from $\mathcal{F}_i$ to $\mathcal{F}$—that is, when each set R_i, which contains all the coherent extensions of P_i from $\mathcal{F}_i$ to $\mathcal{F}$, is just the singleton $\{P_i^\star\}$. In such a case, (DI) tell us, you pick the aggregation function you favour for those cases in which $\mathcal{F}_1 = \ldots = \mathcal{F}_n$, and you apply it to the extended credence functions $P_1^\star, \ldots, P_n^\star$, which are all defined on $\mathcal{F}$.

As it stands, however, (DI) does not tell us how to proceed when, for some individual i, there is more than one coherent credence function that extends P_i from $\mathcal{F}_i$ to $\mathcal{F}$—that is, when some R_i contains more than one credence function. As we will see, in such cases, how we should proceed depends on the purpose for which we are aggregating the individuals' opinions. These purposes can be divided, very roughly, into two sets, which we might describe as *descriptive* and *normative*. In descriptive cases, we aggregate the individuals' credences in order to provide a compressed summary description of their opinions. It is a situation in which we might be better served by simply providing a compendium of all the individuals' credences. However, because of constraints either on the time we have available to present their opinions, or on the storage space of whatever device we use to record them, or on the attention span or cognitive power of the person receiving the information, we need a summary. I treat this case in Section 9.1. In normative cases, we aggregate the credences not so much to provide a compressed description of the individual members' descriptions, but rather to provide an account of the group's opinion, where the group is in some sense treated as an entity in its own rights. I treat this case in Section 9.2.

9.1 Aggregating credences for descriptive purposes

Let's begin with the descriptive aim of aggregating credences. From the examples given in the introduction, these are the ones I envisage falling into this categories: the demographic modellers whose views we wish to summarise for policymakers; the ice sheet experts whose credences we wish to combine to provide the probabilities for different sea level rises we will use in our models of the whole Earth system; the superforecasters whose opinions we wish to aggregrate to inform give our own opinions and decision-making. In each of these cases, we essentially use the individuals in these groups as a sort of instrument or measuring device to provide information about the world.

There are then two sorts of case: either each individual in fact assigns credences to all of the propositions in $\mathcal{F}$, but for each individual i, we only know the credence they assign to the propositions in $\mathcal{F}_i$; or each individual i genuinely only assigns credences to the propositions in $\mathcal{F}_i$. In the former case, the credences we seek exist, but we don't know what they are; in the latter case, they do not exist, but we might suppose that there is some determinate fact about what they *would* be *were* each individual to consider all the propositions in $\mathcal{F}$ and assign credences to them.

In either case, the problem is simply a particular case of a standard problem, namely, how to estimate a quantity. In the first case, each individual assigns credences to all the propositions in $\mathcal{F}$, but we don't know all of them. So the linear pool and geometric pool and other aggregates of those credence functions over $\mathcal{F}$ are determined, but we don't know what they are. So we place a probability function over the possible values it might take in a way that captures our uncertainty about it, and we take the expected value of these possible aggregates to be the aggregate credence we report. In the second case, each individual assigns credences only to the propositions in $\mathcal{F}_i$, but there is some determinate fact about what credence they *would* assign to the remaining propositions in $\mathcal{F}$ *were they to consider them*. So, again, the linear pool and geometric pool and other aggregates of those credence functions over $\mathcal{F}$ are determined, but we don't know what they are. So, again, we place a probability function over the possible values it might take in a way that captures our uncertainty about it, and we take the expected value of these possible aggregates to be the aggregate credence we report.

9.2 Aggregating credences for normative purposes

Let's turn next to the more difficult case in which we aggregate not because we think of the individuals as instruments from which we can learn about the world, but because the group to which they belong plays a role in some normative enterprise and we wish to discover what that group thinks for that normative purpose. From the examples given in the introduction, these are the ones I envisage falling into this category: the epidemiologists of viruses whose views we wish to present as the view of the scientific community in our textbooks; the co-authors on a multi-authored scientific paper whose aggregate view as a collective author we wish to present to the scientific community; and the employees of a company or institution whose collective view we wish to identify in order to assess liability for some harm. In the first two cases, the normative enterprise in which the groups play are role is the enterprise of science, which has norms that govern the assertions included in textbooks and scientific papers. In the third case, the normative enterprise is the legal system, and there are norms here that govern the beliefs we ascribe to an individual whose liability for some harm we are assessing.

While these normative enterprises and the roles within them that the groups play are quite different, a similar norm governs how we should extend each individual i's credences from $\mathcal{F}_i$ to $\mathcal{F}$. It is a conservative norm. It says that we should extend them in the most conservative and unopinionated way possible. That is, we should introduce as little in the way of further opinions as we can when we extend.

Why is this the appropriate norm in the scientific case? In fact, I think there are two reasons. The first reason is the duty of the textbook's author or the paper's lead author to represent fairly the views of the individuals on behalf of whom they write. The textbook's author presents the views of that part of the scientific community; the lead author presents the views of their fellow co-authors. In both cases, they have a duty not to impute to those individuals any further opinions beyond what is necessary to extend their credences to the full set $\mathcal{F}$. The second reason is the duty of scientific authors to their audience. Now, I don't think it is the duty of each scientist not to form opinions beyond what is strictly implied by their evidence. Over years of training and experience in their field, scientists gain an ability to form opinions on the basis of the evidence that sometimes seems to go beyond what the non-expert might conclude, and yet which it is legitimate to report in a scientific publication because of the expertise of the scientist. Nonetheless, when the scientist hasn't formed any opinion about a proposition and when we must nonetheless ascribe an opinion to them in order to carry out the aggregation, we are obliged to make that opinion as conservative as possible. In other words, deviations from a sort of Cliffordian conservatism about opinion are permitted, but only when they are made explicitly by the scientist, and not when they are made by a textbook author or lead author on a paper who is filling in the gaps in another scientist's opinions.

Why is conservatism the appropriate norm in the legal case? Here, I think the key lies in the legal notion of the 'reasonable person'. Often this abstract individual is invoked to personify a certain standard of proof that is required in order to find a defendant liable or guilty. On the websites of many US police departments, you will find a definition of 'probable cause' in terms of what a 'reasonable person' would believe on the basis of the evidence in hand. But it is also used to determine when a defendant's actions are reasonable. For instance, in *Brown vs. Kendall*, Chief Justice Shaw determined that the 'ordinary care' that is necessary for the defendant to avoid liability is "that kind and degree of care, which prudent and cautious men would use".[5] And in *Commonwealth vs. Horsfall*, Chief Justice Rugg declared that "every traveller upon a highway is bound to exercise the care of the ordinarily prudent and cautious person under all circumstances".[6] In both of these cases, we see that the 'reasonable person' is identified with the 'prudent and cautious person'. In the cases cited, the prudence and caution relate to the individual's practical choices about their actions; but it seems reasonable to infer that the same condition is placed on the individual's beliefs. Take the case of *Commonwealth vs. Horsfall*, where a car on a public highway hit an

[5] *Brown vs. Kendall*, 60 Mass. 292 (1850).

[6] *Commonwealth vs. Horsfall*, 213 Mass. 232 (1913).

individual, who then died from their injuries. The individual who was killed was stationary, and the driver had seen them from some distance off and sounded their horn. While there was plenty of room to pass, the driver didn't take it, presumably thinking that the person would move out of the way at the sound of the horn. Even if it might have been rationally permissible to have a high credence that the person would move out of the way, given the driver's evidence, if that high credence is unusually high or incautious or unreasonable, it seems that its rationality would not exculpate them. Rather, when they are assessed for liability, their action is assessed from the point of view of a person who is cautious in both their beliefs and the actions they perform on the basis of those beliefs.

Now let me explain how we might respect these conservative norms formally. If we wish to extend a credence function in the most conservative way possible, it's natural to appeal to the Principle of Maximum Entropy. Typically, that principle applies to an individual whose evidence constrains their credences to some extent, but still permits a range of different credence functions. It is then used to pick out a single credence function from among those: it picks the one that has maximal Shannon entropy.[7] The idea is this: Shannon entropy measures how unopinionated a probability distribution is. The higher its entropy, the less opinionated it is. Thus, a uniform distribution over a finite partition, which is maximally unopinionated, receives the highest entropy among probability functions over that partition, while a probability function that places all of its mass on a single possible world, and is therefore maximally opinionated, receives the lowest entropy. The idea is that your credence function should respect your evidence; but among credence functions that do this, it should be the least opinionated. In this sense, it should not go beyond the evidence; it should not encode opinions that aren't demanded by the evidence.

In our case, the situation is a little different. It is not only the individual's evidence that constrains how we might extend their credences to the propositions that lie in $\mathcal{F}$ but not in $\mathcal{F}_i$. It is also the credences that they assign to the propositions in $\mathcal{F}_i$. So we might imagine that each individual has their own body of evidence $\mathbf{E}_i$, and we might model this as the set of credence functions on $\mathcal{F}$ that respect that evidence. Thus, for instance, if among individual i's body of evidence is the fact that the coin in their pocket is fair, then each credence function in $\mathbf{E}_i$ should assign credence 50% to that coin landing heads if tossed; and so on. Now, just as we are supposing that all individuals have coherent credence functions, so we might suppose that they all have credence functions that respect their evidence. Thus, for all i, R_i and $\mathbf{E}_i$ overlap. Then we might say: when we extend individual i's credence function from $\mathcal{F}_i$ to $\mathcal{F}$, we should ascribe the credence function P_i^{ME}, which is defined on $\mathcal{F}$ as follows:

$$P_i^{\mathrm{ME}} = \underset{P \in \mathbf{E}_i \cap R_i}{\arg\max}\, H(P)$$

where, recall:

- $\mathbf{E}_i$ is the set of credence functions on $\mathcal{F}$ that respect the evidence that individual i has;
- R_i is the set of coherent credence functions on $\mathcal{F}$ that extend P_i; and
- $H(P)$ is the Shannon entropy of P.

The motivation is the same as in the standard application of maximal entropy reasoning, where an individual's credences are constrained only by their evidence, and we demand that

[7] It is typically used only when it is guaranteed that there will be just one such credence function.

they pick among those that satisfy the constrains the one that is least opinionated. Similarly here, where both the individual's evidence and their existing credences impose constraints, we ascribe to them the credence function among those that satisfies both constraints that is least opinionated. Thus, we define

$$\Delta_{\mathrm{ME}^*}(P_1, P_2) = \Delta(P_1^{\mathrm{ME}}, P_2^{\mathrm{ME}})$$

where Δ is our favoured aggregation function for credence functions defined on the same set of propositions—e.g., linear pooling (Δ_{LP}) or geometric pooling (Δ_{GP}).

Figure 3 illustrates the result of this process in the case we've considered before where:

- each individual has no evidence, so that $\mathbf{E}_1 = \mathbf{E}_2 = \mathcal{P}_{\mathcal{F}}$; and
- the propositions X, Y, and Z form a partition and the individuals' credences are as follows:

	X	Y	Z
P_1	0.8	–	–
P_2	–	0.8	–

Then

	X	Y	Z
P_1^{ME}	0.8	0.1	0.1
P_2^{ME}	0.1	0.8	0.1

Then, if Δ_{LP} is linear pooling, then

	X	Y	Z
$\Delta^{\mathrm{LP}}_{\mathrm{ME}^*}(P_1, P_2)$	0.45	0.45	0.1

It's worth noting that, when we combine this with the illustration from above, we see that taking the credence function that maximises entropy among all linear pools of the possible extensions of P_1 and P_2 is not the same as taking the linear pool of the extensions of P_1 and P_2 that maximise entropy. That is, $\Delta^{\mathrm{LP}}_{\mathrm{ME}}(P_1, P_2) \neq \Delta^{\mathrm{LP}}_{\mathrm{ME}^*}(P_1, P_2)$. And, it seems to me at least, the latter gives the more sensible result.

10 Conclusion

Often in philosophy, when we ask how we should do something, we must specify the purpose for which we're doing it. It turns out that this is true in the case of aggregating the probabilistic judgments of individuals in a group when not all of those individuals assign probabilities to the same propositions. In this paper, I ended up discussing two such purposes. First, those cases in which you use the individuals in the group and the group itself as a sort of instrument or measuring device through which to gain information about the world. In these cases, the solution is straightforward and a particular case of the more general problem of determining the reading of an instrument when you don't have full information about it. Second, those cases in which the group that the individuals constitute plays an important role in some normative enterprise, such as the enterprise of science or as a player whose liability for some harm you are assessing. In these cases, the solution is less straightforward, but I argued that, in both of the sorts of cases mentioned, we are obliged to extend the credences of each individual in the group in the more conservative way possible.

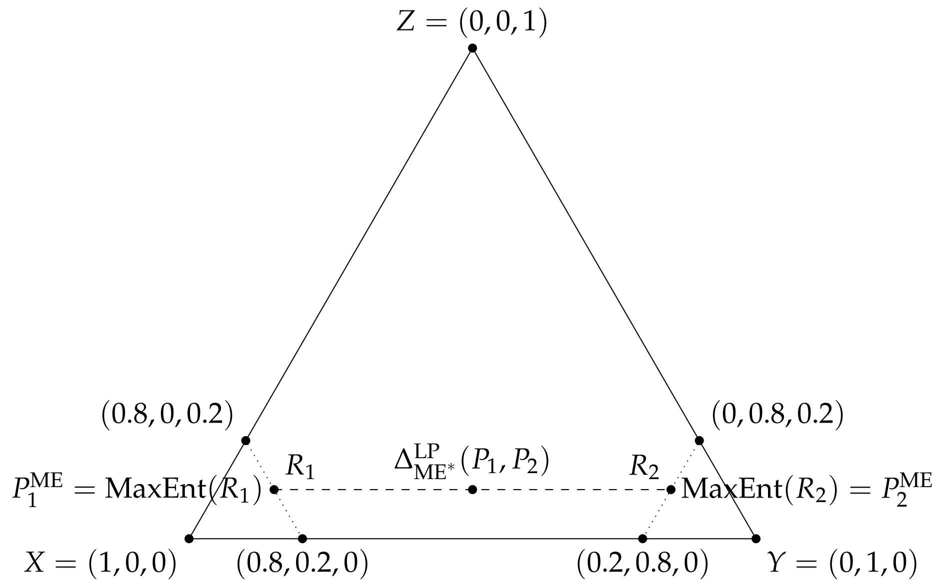

Figure 3: The barycentric plot of the simplex with $(1, 0, 0)$ at bottom right, $(0, 1, 0)$ at bottom left, and $(0, 0, 1)$ at the top.

11 Appendix: proofs

Proposition 1 *If $\eth$ is differentiable in its first argument, CAP$_\eth$ violates (DI).*

Proof. Suppose $W = \{w_1, w_2\}$ and $\mathcal{F}_1 = \{w_1, w_2\}$ and $\mathcal{F}_2 = \{w_2\}$. Then (DI) says that, for any P_1 defined on $\mathcal{F}_1$ and P_2 defined on $\mathcal{F}_2$,

$$\eth(x, P_1(w_1)) + \eth(1 - x, P_1(w_2)) + \eth(1 - x, P_2(w_2))$$

is minimized as a function of x at $x = p$ iff

$$\eth(x, P_1(w_1)) + \eth(1 - x, P_1(w_2)) + \eth(x, P_2(w_1)) + \eth(1 - x, 1 - P_2(w_1))$$

is minimized as a function of x at $x = p$. Now, differentiate each with respect to x and evaluate at p, where it will take value 0:

$$\left[\frac{d}{dx}\eth(x, P_1(w_1)) + \eth(1 - x, P_1(w_2)) + \eth(1 - x, P_2(w_2))\right](p) =$$
$$\eth'(p, P_1(w_1)) - \eth'(1 - p, P_1(w_2)) - \eth'(1 - p, P_2(w_2)) = 0$$

And

$$\left[\frac{d}{dx}\eth(x, P_1(w_1)) + \eth(1 - x, P_1(w_2)) + \eth(x, P_2(w_1)) + \eth(1 - x, P_2(w_2))\right](p) =$$
$$\eth'(p, P_1(w_1)) - \eth'(1 - p, P_1(w_2)) + \eth'(p, P_2(w_1)) - \eth'(1 - p, P_2(w_1)) = 0$$

But subtracting the first from the second, we get:

$$\mathfrak{d}'(p, P_2(w_1)) = 0$$

But, since $\mathfrak{d}$ is a divergence, $\mathfrak{d}(x, P_2(w_1))$ is minimized, as a function of x, uniquely at $x = P_2(w_1)$. So $p = P_2(w_1)$. But by similar reasoning, we can also establish:

$$\mathfrak{d}'(p, P_1(w_1)) = 0$$

So it is minimized at $p = P_1(w_1)$. But if $P_1(w_1) \neq P_2(w_1)$, then this gives a contradiction. $\square$

Theorem 4 is a corollary of this:

Theorem 5. *Suppose $f_1, \ldots, f_m, g_1, \ldots, g_m$ are linear functions in n variables. That is, for each $1 \leq j \leq m$, there are $\alpha_{j1}, \ldots, \alpha_{jn}, \beta_{j1}, \ldots, \beta_{jn}$ such that*

$$f_j(x_1, \ldots, x_n) = \alpha_{j1}x_1 + \ldots + \alpha_{jn}x_n + k_j$$

and

$$g_j(x_1, \ldots, x_n) = \beta_{j1}x_1 + \ldots + \beta_{jn}x_n + l_j$$

And suppose $\mathbf{x} = x_1, \ldots, x_n$ minimizes

$$\sum_j (f_j(\mathbf{x}) - f_j(\mathbf{p}))^2 + \sum_j (f_j(\mathbf{x}) - f_j(\mathbf{q}))^2 + \sum_j (g_j(\mathbf{x}) - g_j(\mathbf{p}))^2 \tag{1}$$

and $\mathbf{y} = y_1, \ldots, y_n$ minimizes

$$\sum_j (f_j(\mathbf{y}) - f_j(\mathbf{p}))^2 + \sum_j (f_j(\mathbf{y}) - f_j(\mathbf{q}))^2 + \sum_j (g_j(\mathbf{y}) - g_j(\mathbf{q}))^2 \tag{2}$$

Then, for $1 \leq i \leq n$, let $z_i = \frac{x_i + y_i}{2}$. Then $\mathbf{z} = z_1, \ldots, z_n$ minimizes

$$\sum_j (f_j(\mathbf{x}) - f_j(\mathbf{p}))^2 + \sum_j (f_j(\mathbf{x}) - f_j(\mathbf{q}))^2 + \sum_j (g_j(\mathbf{x}) - g_j(\mathbf{p}))^2 + \sum_j (g_j(\mathbf{x}) - g_j(\mathbf{p}))^2 \tag{3}$$

Proof. Suppose $\mathbf{x}$ minimizes (1). Then, for all $1 \leq k \leq n$,

$$\sum_j 2\alpha_{jk} \left(\sum_i \alpha_{ji}(x_i - p_i) \right) + \sum_j 2\alpha_{jk} \left(\sum_i \alpha_{ji}(x_i - q_i) \right) + \sum_j 2\beta_{jk} \left(\sum_i \beta_{ji}(x_i - p_i) \right) = 0$$

And suppose $\mathbf{y}$ minimizes (2). Then, for all $1 \leq k \leq n$,

$$\sum_j 2\alpha_{jk} \left(\sum_i \alpha_{ji}(y_i - p_i) \right) + \sum_j 2\alpha_{jk} \left(\sum_i \alpha_{ji}(y_i - q_i) \right) + \sum_j 2\beta_{jk} \left(\sum_i \beta_{ji}(y_i - q_i) \right) = 0$$

Now, it's easy to check that, if we let $z_i = \frac{p_i+q_i}{2}$, then $\mathbf{z}$ minimizes (3). So now we need only show that $\frac{x_i+y_i}{2} = \frac{p_i+q_i}{2}$. That is, $x_i = p_i + q_i - y_i$. So, let $x_i^* = p_i + q_i - y_i$. Then

$$\sum_j 2\alpha_{jk}\left(\sum_i \alpha_{ji}(x_i^* - p_i)\right) + \sum_j 2\alpha_{jk}\left(\sum_i \alpha_{ji}(x_i^* - q_i)\right) + \sum_j 2\beta_{jk}\left(\sum_i \beta_{ji}(x_i^* - p_i)\right) =$$

$$\sum_j 2\alpha_{jk}\left(\sum_i \alpha_{ji}((p_i + q_i - y_i) - p_i)\right) + \sum_j 2\alpha_{jk}\left(\sum_i \alpha_{ji}((p_i + q_i - y_i) - q_i)\right) + \sum_j 2\beta_{jk}\left(\sum_i \beta_{ji}((p_i + q_i - y_i) - p_i)\right) =$$

$$\sum_j 2\alpha_{jk}\left(\sum_i \alpha_{ji}(q_i - y_i)\right) + \sum_j 2\alpha_{jk}\left(\sum_i \alpha_{ji}(p_i - y_i)\right) + \sum_j 2\beta_{jk}\left(\sum_i \beta_{ji}(q_i - y_i)\right) = 0$$

Since $\mathbf{y}$ minimizes (2). So $\mathbf{x}^*$ minimizes (1). So $\mathbf{x}^* = \mathbf{x}$, and $x_i = x_i^* = p_i + q_i - y_i$, as required. □

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